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Mean Convergence of Generalized Jacobi Series and Interpolating Polynomials, II

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Weighted mean convergence of interpolating polynomials based on the zeros of generalized Jacobi polynomials is investigated. The approach is based on generalized Jacobi series and Marcinkiewicz–Zygmund type inequality. © 1994 Academic Press, Inc.

1. INTRODUCTION

The purpose of this paper is to investigate weighted mean convergence of interpolating polynomials based on the zeros of generalized Jacobi polynomials.

Our approach is based on the weighted mean convergence of generalized Jacobi series, which is studied in Part I of this paper [30]. The main result of [30] gives an inequality of the type

$$\|S_n(f)\, U\|_p \leqslant c\, \|fV\|_p, \tag{1.1}$$

where $S_n(f)$ is the nth partial sum of generalized Jacobi series of f, U and V are suitable weight functions, and c is a constant independent of n and f. In this part, we use (1.1) to prove an inequality of the type

$$\int_{-1}^{1} |P|^p\, w\, dx \leqslant c \sum_{k=1}^{n} |P(x_{kn})|^p\, \lambda_{kn}, \qquad 1 < p < +\infty, \tag{1.2}$$

where P is a polynomial of degree at most $n-1$, x_{kn} are the zeros of orthogonal polynomial $p_n(w, x)$, and c is a constant independent of n and P. An inequality of this type is called a Marcinkiewicz–Zygmund type

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inequality. It enables us to prove the mean convergence of the Lagrange interpolation. See Part I for the background and history.

After Nevai's article [15], in which he found necessary and sufficient conditions for weighted mean convergence of Lagrange interpolation based on the zeros of the generalized Jacobi polynomials, the weighted mean convergence of various other interpolating polynomials have been studied, including the Hermite and Hermite–Fejér interpolating polynomials. However, the method in these works (see [17, 18, 24, 25]) requires detailed information on the explicit formulas of these interpolating polynomials, and various terms in these formulas and certain quadrature sums have to be estimated very carefully. So far only those interpolating polynomials based on the zeros of the Jacobi polynomials (or slightly general ones) have been successfully handled, and it seems very difficult to apply this mehod to the more general cases. As an alternative approach, we show that the inequality (1.2) can be extended to the polynomials of degree at most $mn-1$ with the right-hand side containing derivative values up to $m-1$ order. Thus we are able to prove the weighted mean convergence of various interpolating polynomials based on the zeros of the generalized Jacobi polynomials. It seems that the method we present is the "right one" for investigating the mean convergence of the interpolating polynomials. Not only is it more powerful, but it is also simpler than the previous method, because we do not need knowledge of explicit formulas in order to apply the method. For the historical account, we refer to [1, 6, 7, 10–15, 17–19, 23–31].

The paper is organized as follows. The next section is devoted to notations and preliminaries. The Marcinkiewicz–Zygmund inequality and its extension are presented in Section 3. The mean convergence of the interpolating polynomials is discussed in Section 4.

2. Preliminaries

Let $d\alpha = \alpha'(x)\,dx$ be a nonnegative distribution on $[-1, 1]$. Let $p_n(d\alpha, x)$ be the sequence of polynomials othonormal with respect to $d\alpha$. The zeros of $p_n(d\alpha)$ are denoted by $x_{kn}(d\alpha)$ and the following order is assumed:

$$1 > x_{1n}(d\alpha) > x_{2n}(d\alpha) > \cdots > x_{nn}(d\alpha) > -1. \tag{2.1}$$

The Christoffel function $\lambda_n(d\alpha)$ is defined by

$$\lambda_n(d\alpha, x) = \left[\sum_{k=0}^{n-1} p_k^2(d\alpha, x)\right]^{-1}. \tag{2.2}$$

The numbers $\lambda_{kn}(d\alpha) = \lambda_n(d\alpha, x_{kn})$ are called the Cotes numbers. By the Gauss–Jacobi quadrature formula [22, p. 47],

$$\sum_{k=1}^{n} P(x_{kn}(d\alpha))\, \lambda_{kn}(d\alpha) = \int_{-1}^{1} P\, d\alpha \tag{2.3}$$

holds for every polynomial $P \in \Pi_{2n-1}$. Here, Π_n is the space of polynomials of degree at most n.

Let $m \geqslant 1$ be a given integer. For a $(m-1)$th differentiable function f, the Hermite interpolating polynomials corresponding to the distribution $d\alpha$, denoted by $H_{mn}(d\alpha, f)$, are defined to be the unique polynomial of degree at most $mn-1$ satisfying

$$H_{mn}^{(j)}(d\alpha, f, x_{kn}) = f^{(j)}(x_{kn}), \qquad 0 \leqslant j \leqslant m-1, \quad 1 \leqslant k \leqslant n, \tag{2.4}$$

where $x_{kn} = x_{kn}(d\alpha)$. When $m = 1$, $H_{mn}(d\alpha, f)$ are the Lagrange interpolating polynomials, we write $L_n(d\alpha, f) = H_{1,n}(d\alpha, f)$.

If $0 < p \leqslant +\infty$, then $f \in L^p$ if $\|f\|_p < +\infty$ where

$$\|f\|_p = \left(\int_{-1}^{1} |f(t)|^p\, dt \right)^{1/p}, \qquad 0 < p < +\infty,$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \in [-1,1]} |f(t)|.$$

Of course, when $0 < p < 1$, $\|\cdot\|_p$ is not a norm; nevertheless, we keep this notation for convenience. We also use the notations $\|\cdot\|_{d\alpha, p}$ and $\|\cdot\|_{w, p}$, defined by

$$\|f\|_{d\alpha, p} = \left(\int_{-1}^{1} |f|^p\, d\alpha \right)^{1/p} \qquad \text{or} \qquad \|f\|_{w, p} = \left(\int_{-1}^{1} |f|^p\, w\, dt \right)^{1/p}, \tag{2.5}$$

even for $0 < p < 1$.

Let w be a nonnegative function. We call w a generalized Jacobi weight function ($w \in GJ$), if it can be written as

$$w(x) = \prod_{i=0}^{r+1} |x - t_i|^{\Gamma_i} \tag{2.6}$$

for $x \in [-1, 1]$ and $w(x) = 0$ for $|x| > 1$. Note that w is not necessarily integrable. We call $d\alpha$ a generalized Jacobi distribution when $\alpha' = \psi w$, where $w \in GJ$ and w is integrable, ψ is a positive continuous function in $[-1, 1]$ and the modulus of continuity ω of ψ satisfies

$$\int_0^1 \frac{\omega(t)}{t}\, dt < +\infty.$$

Sometimes we write $\Gamma_i(w)$ or $\Gamma_i(d\alpha)$ in place of Γ_i to indicate that they are parameters of w or $d\alpha$, respectively. Orthogonal polynomials corresponding to generalized Jacobi distribution are called generalized Jacobi polynomials. When $\Gamma_i = 0$, $1 \leqslant i \leqslant r$, and $\psi = 1$, "generalized Jacobi" reduces to "Jacobi."

Throughout this paper, we use letters $c, c_1, c_2, \ldots$, etc., to denote constants depending only on weight functions and other fixed parameters involved, but their values may be different at different occurrences, even within the same formula. The notation $A \sim B$ means $|A^{-1}B| \leqslant c$ and $|AB^{-1}| \leqslant c$.

Let $d\alpha$ be a generalized Jacobi distribution. Let $S_n(d\alpha, f)$ be the partial sum of the generalized Jacobi series, i.e.,

$$S_n(d\alpha, f, x) = \sum_{k=0}^{n-1} c_k(f)\, p_k(d\alpha, x),$$

where

$$c_k(f) = \int_{-1}^{1} f(x)\, p_k(d\alpha, x)\, d\alpha.$$

The main result in [30], which is essential for this paper, is the following.

THEOREM 2.1. *Let $d\alpha$ be a generalized Jacobi distribution, and let u and w be generalized Jacobi weight functions. Let $1 < p < +\infty$. Then*

$$\|S_n(d\alpha, f)\, w\|_{d\alpha, p} \leqslant c\, \|fu\|_{d\alpha, p} \tag{2.8}$$

for every f such that $\|fu\|_{d\alpha, p} < +\infty$ if and only if

$$\begin{aligned} &w^p \alpha' \in L^1, && u^{-q} \alpha' \in L^1 \\ &w^p(\alpha' \sqrt{1-x^2})^{-p/2}\, \alpha' \in L^1, && u^{-q}(\alpha' \sqrt{1-x^2})^{-q/2}\, \alpha' \in L^1 \end{aligned} \tag{2.9}$$

and

$$w(x) \leqslant cu(x). \tag{2.10}$$

In the following, we list those properties of the generalized Jacobi polynomials that are used in this article. For the proof of these properties and the extensive study of generalized Jacobi polynomials, see [2] and [14]. For $w \in GJ$ in the form of (2.6) we define

$$w_n(x) = \left(\sqrt{1-x} + \frac{1}{n}\right)^{2\Gamma_0} \prod_{i=1}^{r} \left(|x - t_i| + \frac{1}{n}\right)^{\Gamma_i} \left(\sqrt{1+x} + \frac{1}{n}\right)^{2\Gamma_{r+1}}. \tag{2.11}$$

For $d\alpha$ being a generalized Jacobi distribution, we also denote the corresponding one for $\alpha' = \psi w$ as $\alpha'_n = \psi w_n$.

LEMMA 2.2. *Let $d\alpha$ be a generalized Jacobi distribution. Then for every positive integer n*

$$|p_n(d\alpha, x)| \leqslant c\alpha'_n(x)^{-1/2}\left(\sqrt{1-x^2}+\frac{1}{n}\right)^{-1/2} \tag{2.12}$$

uniformly for $-1 \leqslant x \leqslant 1$ [2, Theorem 1.1, p. 226],

$$\lambda_n(d\alpha, x) \sim \frac{1}{n}\alpha'_n(x)\left(\sqrt{1-x}+\frac{1}{n}\right)\left(\sqrt{1+x}+\frac{1}{n}\right) \tag{2.13}$$

uniformly for $-1 \leqslant x \leqslant 1$, *in particular*

$$\begin{aligned}\lambda_{kn}(d\alpha) \sim \frac{1}{n}(1-x_{kn})^{\Gamma_0(d\alpha)+1/2}\prod_{i=1}^{r}\left(|t_i - x_{kn}|+\frac{1}{n}\right)^{\Gamma_i(d\alpha)}\\ \times(1+x_{kn})^{\Gamma_{r+1}(d\alpha)+1/2}\end{aligned} \tag{2.14}$$

uniformly for $1 \leqslant k \leqslant n$, *where* $x_{kn} = x_{kn}(d\alpha)$, *and* [14, p. 170]

$$|p'_n(d\alpha, x_{kn})|^{-1} \sim \frac{1}{n}[\alpha'_n(x_{kn})]^{1/2}(1-x_{kn}^2)^{3/4} \tag{2.15}$$

uniformly for $1 \leqslant k \leqslant n$.

Let $w \in GJ$, for a fixed $d > 0$, we define $\Delta_n(d)$ by

$$\Delta_n(d) = [-1+dn^{-2}, 1-dn^{-2}]\Big\backslash \bigcup_{i=1}^{r}[t_i - dn^{-1}, t_i + dn^{-1}].$$

We use χ_E to denote the characteristic function of a set E.

LEMMA 2.3. [14, Theorem 6.3.28, p. 120]. *Let $w \in GJ$ be integrable. Then for each $0 < p < +\infty$ there exists $d = d(p) > 0$ such that for every $R \in \Pi_n$,*

$$\|R\|_{w,p} \leqslant c\,\|R\chi_{\Delta_n(d)}\|_{w,p}.$$

3. THE MARCINKIEWICZ–ZYGMUND TYPE INEQUALITIES

We prove the Marcinkiewicz–Zygmund type inequalities in this section. The essential one is presented in Theorem 3.2 and extension of it in Theorem 3.3. First we state one lemma which is used in our proof.

LEMMA 3.1. *Let $d\alpha$ be a generalized Jacobi distribution and $u \in GJ$. Let $m \geqslant 1$, $P \in \Pi_{mn-1}$ and $1 < p < +\infty$. Then*

$$\sum_{k=1}^{n} |P(x_{kn}(d\alpha))|^p \, u_n(x_{kn}(d\alpha)) \, \lambda_{kn}(d\alpha) \leqslant c \int_{-1}^{1} |P(t)|^p \, u(t) \, d\alpha, \tag{3.1}$$

where u_n is defined at (2.11), *and c depends on m and p.*

This lemma is proved in [14, Theorem 9.25, p. 169]; see also [6, Theorem 5].

THEOREM 3.2. *Let $P \in \Pi_{n-1}$ and $1 < p < +\infty$. Let $d\alpha$, $d\beta$ be generalized Jacobi distributions and $u \in GJ$, such that*

$$u^{1-q}\alpha' \in L^1, \qquad u^{1-q}(\alpha' \sqrt{1-x^2})^{-q/2} \, \alpha' \in L^1, \tag{3.2a}$$

$$u\alpha' \geqslant c\beta', \tag{3.2b}$$

and

$$(\alpha' \sqrt{1-x^2})^{-p/2} \, \beta' \in L^1. \tag{3.2c}$$

Then

$$\|P\|_{d\beta, p} \leqslant c \left(\sum_{k=1}^{n} |P(x_{kn}(d\alpha))|^p \, u_n(x_{kn}(d\alpha)) \, \lambda_{kn}(d\alpha) \right)^{1/p}, \tag{3.3}$$

where u_n is defined at (2.11).

Proof. We write x_{kn} for $x_{kn}(d\alpha)$ in the following. We have

$$\|P\|_{d\beta, p} = \sup_{\|g\|_{d\beta, q} = 1} \int_{-1}^{1} P(t) \, g(t) \, d\beta.$$

By the orthogonality, the Gauss–Jacobi quadrature (2.3), the Hölder inequality, and Lemma 3.1, we have

$$\begin{aligned} \int_{-1}^{1} P(t) \, g(t) \, d\beta &= \int_{-1}^{1} P(t) \, S_n(d\alpha, g\beta'\alpha'^{-1}, t) \, d\alpha \\ &= \sum_{k=1}^{n} P(x_{kn}) \, S_n(d\alpha, g\beta'\alpha'^{-1}, x_{kn}) \, \lambda_{kn}(d\alpha) \\ &\leqslant c \left(\sum_{k=n}^{n} |P(x_{kn})|^p \, \lambda_{kn}(d\alpha) \, u_n(x_{kn}) \right)^{1/p} \\ &\quad \times \|S_n(d\alpha, g\beta'\alpha'^{-1}) \, u^{-1/p}\|_{d\alpha, q}. \end{aligned}$$

We now apply Theorem 2.1 with $g\beta'\alpha'^{-1}$, $u^{-1/p}$, $(\beta'^{-1}\alpha')^{1/p}$, and q in place of f, w, u, and p, and conclude that

$$\|S_n(d\alpha, g\beta'\alpha'^{-1})\, u^{-1/p}\|_{d\alpha, q} \leqslant c\, \|g\|_{d\beta, q}.$$

The conditions (3.2) become conditions (2.9) and (2.10) under this substitution. The proof is completed. ∎

The conditions (3.2) in this theorem seem to be quite complicated at the first glance. They become even more complicated in the more general cases discussed in Theorem 3.3. However, as we show later, a special choice of the auxiliary weight u reduces these conditions to a sole condition (3.2c).

To state our extensions of Theorem 3.2, we need the following definitions. Let $d\alpha$ be a generalized Jacobi distribution. Associated with $d\alpha$, we define a generalized Jacobi weight v by

$$\begin{aligned} &\Gamma_0(v) = \min\{0, \Gamma_0(d\alpha) + \tfrac{1}{2}\}, \qquad \Gamma_{r+1}(v) = \min\{0, \Gamma_{r+1}(d\alpha) + \tfrac{1}{2}\} \\ &\Gamma_i(v) = \min\{0, \Gamma_i(d\alpha)\}, \qquad 1 \leqslant i \leqslant r. \end{aligned} \tag{3.4}$$

We also use v^* to denote

$$v^*(x) = (\alpha' \sqrt{1-x^2})\, v^{-1}(x). \tag{3.5}$$

From these definitions, it readily follows that

$$v^{-1}(x) \leqslant c \qquad \text{and} \qquad v^*(x) \leqslant c. \tag{3.6}$$

THEOREM 3.3. *Let $m \geqslant 1$, $P \in \Pi_{mn-1}$, and $1 < p < +\infty$. Let $d\alpha$, $d\beta$ be generalized Jacobi distributions and $u \in GJ$, such that $u\alpha' \in L^1$ and*

$$u^{1-q} v^{(m-1)q/2} \alpha' \in L^1, \qquad u^{1-q} v^{(m-1)q/2} (\alpha' \sqrt{1-x^2})^{-q/2}\, \alpha' \in L^1, \tag{3.7a}$$

$$u\alpha' \geqslant c\beta'(v^*)^{-(m-1)q/2}, \tag{3.7b}$$

and

$$(\alpha' \sqrt{1-x^2})^{-mp/2}\, \beta' \in L^1. \tag{3.7c}$$

Then

$$\|P\|_{d\beta, p} \leqslant c \left(\sum_{j=0}^{m-1} \sum_{k=1}^{n} |(\sqrt{1-x_{kn}^2})^j\, P^{(j)}(x_{kn})|^p \times (v_n^*(x_{kn}))^{jp/2}\, u_n(x_{kn})\, \lambda_{kn}(d\alpha)/n^{jp} \right)^{1/p}, \tag{3.8}$$

where $x_{kn} = x_{kn}(d\alpha)$, u_n and v_n^* are defined as in (2.11).

Proof. We use induction. The case $m=1$ reduces to Theorem 3.2. We write $(3.7)_m$ to denote the dependency of (3.7) on m. Suppose the theorem has been proved for polynomials in $\Pi_{(m-1)n-1}$, where $m \geqslant 2$. Let now $P \in \Pi_{mn-1}$. We first note that $(3.7)_m$ implies $(3.7)_{m-1}$. Indeed, by (3.6), the only condition that needs to be checked is $(3.7\text{c})_{m-1}$. Let $1 \leqslant i \leqslant r$, and $\bar{t}_i = (t_1 + t_{i+1})/2$. If $\Gamma_i(d\alpha) < 0$, then both $(\alpha' \sqrt{1-x^2})^{-(m-1)p/2}$ and $(\alpha' \sqrt{1-x^2})^{-mp/2}$ are integrable on $[\bar{t}_{i-1}, \bar{t}_i]$, and if $\Gamma_i(d\alpha) \geqslant 0$, then $\underline{\alpha' \leqslant c}$ on $[\bar{t}_{i-1}, \bar{t}_i]$; thus $(\alpha' \sqrt{1-x^2})^{-(m-1)p/2}$ is dominated by $(\alpha' \sqrt{1-x^2})^{-mp/2} \beta'$. Similarly for $i=0$ and $i=r+1$. Thus $(3.7\text{c})_m$ implies $(3.7\text{c})_{m-1}$.

It is known that $H_{mn}(d\alpha, P, x) = P(x)$; thus from (2.4) we have

$$P(x) - H_{n,m-1}(d\alpha, P, x) = p_n^{m-1}(d\alpha, x)\, Q_n(x), \tag{3.9}$$

where $Q_n \in \Pi_{n-1}$. Since it follows from (2.12) that

$$|p_n(d\alpha, x)| \leqslant c(\alpha'(x) \sqrt{1-x^2})^{-1/2}, \qquad x \in \Delta_n(d),$$

we then have by Lemma 2.3

$$\begin{aligned} \|P - H_{n,m-1}(d\alpha, P)\|_{d\beta,p}^p &\leqslant c\, \|(\alpha' \sqrt{1-x^2})^{-(m-1)p/2} Q_n\|_{d\beta,p}^p \\ &\leqslant c \sum_{k=1}^{n} |Q_n(x_{kn})|^p (v_n(x_{kn}))^{-(m-1)p/2} \\ &\quad \times u_n(x_{kn})\, \lambda_{kn}(d\alpha), \end{aligned} \tag{3.10}$$

where the last inequality follows from Theorem 3.2 with $(\alpha' \sqrt{1-x^2})^{-(m-1)p/2} \beta'$ in place of β' and $uv^{-(m-1)p/2}$ in place of u. From (3.9) we get that

$$Q_n(x_{kn}) = \frac{P^{(m-1)}(x_{kn}) - H_{n,m-1}^{(m-1)}(d\alpha, P, x_{kn})}{(m-1)!\, [p_n'(d\alpha, x_{kn})]^{m-1}}.$$

Thus by (2.15), we can estimate the sum in (3.10) by two sums. The first one is

$$\begin{aligned} &\frac{1}{n^{(m-1)p}} \sum_{k=1}^{n} |(\sqrt{1-x_{kn}^2})^{m-1} P^{(m-1)}(x_{kn})|^p \\ &\qquad \times (v_n^*(x_{kn}))^{(m-1)p/2} u_n(x_{kn})\, \lambda_{kn}(d\alpha), \end{aligned}$$

where $v_n^* = (v^*)_n$, which is the $j = m-1$ term in the right-hand side of (3.8). The second one is

$$\frac{1}{n^{(m-1)p}} \sum_{k=1}^{n} |(\sqrt{1-x_{kn}^2})^{m-1} H_{n,m-1}^{(m-1)}(d\alpha, P, x_{kn})|^p \times (v_n^*(x_{kn}))^{(m-1)p/2} u_n(x_{kn}) \lambda_{kn}(d\alpha),$$

which, by Lemma 3.1 with $(v^*)^{(m-1)p/2} u(\sqrt{1-x^2})^{m-1}$ in place of u, is bounded by

$$\frac{c}{n^{(m-1)p}} \|(\sqrt{1-x^2})^{m-1} H_{n,m-1}^{(m-1)}(d\alpha, P)(v^*)^{(m-1)/2} u^{1/p}\|_{d\alpha,p}^p$$
$$\leq c \|H_{n,m-1}(d\alpha, P)(v^*)^{(m-1)/2} u^{1/p}\|_{d\alpha,p}^p,$$

where the second inequality follows from the Bernstein–Markov inequality (cf. [16, Theorem 5]). Since $H_{n,m-1}(d\alpha, P) \in \Pi_{(m-1)n-1}$, we have by induction with $(v^*)^{(m-1)p/2} u\alpha'$ in place of β' that this term is bounded by the right-hand side of (3.8) with $m-1$ replaced by $m-2$. The conditions $(3.7)_{m-1}$ under this substitution are implied by $(3.7)_m$ and $u\alpha' \in L^1$. Thus, we have proved that $\|P - H_{n,m-1}(d\alpha, P)\|_{d\beta,p}$ is bounded by the right-hand side of (3.8). The desired inequality (3.8) now follows from

$$\|P\|_{d\beta,p} \leq \|P - H_{n,m-1}(d\alpha, P)\|_{d\beta,p} + \|H_{n,m-1}(d\alpha, P)\|_{d\beta,p}$$

and the induction. The proof is completed. ∎

Since the Hermite interpolation by polynomials is regular, we can find a nonzero polynomial of degree $> mn-1$ such that the right-hand side of (3.8) is equal to zero. Theorefore, the degree of polynomials in this theorem cannot be greater than $mn-1$.

We now choose a special u in Theorem 3.3 to reduce the conditions. The result is the following Theorem 3.4, which will be used in proving the mean convergence of interpolating polynomials. We need the following conditions on $d\alpha$,

$$\Gamma_0(d\alpha) > -\frac{1}{2} - \frac{1}{m+1}, \qquad \Gamma_i(d\alpha) > \frac{-2}{m+1}, \quad 1 \leq i \leq r,$$
$$\Gamma_{r+1}(d\alpha) > -\frac{1}{2} - \frac{1}{m+1}. \tag{3.11}$$

THEOREM 3.4. *Let $m \geq 1$, $P \in \Pi_{mn-1}$, and $1 < p < +\infty$. Let $d\alpha$, $d\beta$ be generalized Jacobi distributions with $d\alpha$ satisfying* (3.11). *If*

$$(\alpha' \sqrt{1-x^2})^{-mp/2} \beta' \in L^1, \tag{3.12}$$

then there is a generalized Jacobi distribution $d\gamma$, *such that* $\gamma'\alpha^{-1} \in GJ$ *and*

$$\|P\|_{d\beta,p} \leqslant c\left(\sum_{j=0}^{m-1}\sum_{k=1}^{n} |(\sqrt{1-x_{kn}^2})^j P^{(j)}(x_{kn})|^p \lambda_n(d\gamma, x_{kn})/n^{jp}\right)^{1/p}, \quad (3.13)$$

where $x_{kn} = x_{kn}(d\alpha)$, *and* $\lambda_n(d\gamma, *)$ *is the Christoffel function at* (2.2).

Proof. By (2.13) and (2.14), if we take $u = \alpha'^{-1}\gamma'$ in Theorem 3.3 and replace $(v_n^*(x_{kn}))^{jp/2}$ by its constant upper bound from (3.6), then inequality (3.8) implies (3.13). We now define our $d\gamma$ such that the conditions (3.7) of Theorem 3.3 reduce to (3.12) for this choice of u. Let v and v^* be defined as in (3.4). We define γ' by

$$\Gamma_i(\gamma') = \min\left\{\Gamma_i(d\alpha) + \frac{m-1}{2}\Gamma_i(v), \frac{m-1}{2}\Gamma_i(v),\right.$$
$$\left.\Gamma_i(d\beta) - \frac{(m-1)p}{2}\Gamma_i(v^*)\right\}, \qquad 1 \leqslant i \leqslant r, \quad (3.14)$$

and

$$\Gamma_i(\gamma') = \min\left\{\Gamma_i(d\alpha) + \frac{m-1}{2}\Gamma_i(v), -\frac{1}{2} + \frac{m-1}{2}\Gamma_i(v),\right.$$
$$\left.\Gamma_i(d\beta) - \frac{(m-1)p}{2}\Gamma_i(v^*)\right\}, \qquad i = 0, \quad r+1. \quad (3.15)$$

First we show that with γ' so defined, $d\gamma = \gamma'\,dx$ is a generalized Jacobi distribution, i.e., γ' integrable. The conditions (3.11) imply that the first two terms in the brackets of (3.14) and (3.15) are both > -1. By considering $\Gamma_i(d\alpha) \geqslant 0$ and $\Gamma_i(d\alpha) < 0$, $1 \leqslant i \leqslant r$, or $\Gamma_i(d\alpha) + \frac{1}{2} \geqslant 0$ and $\Gamma_i(d\alpha) + \frac{1}{2} < 0$, $i = 0$, $r+1$, separately, we have form (3.12) that $\Gamma_i(d\beta) - (m-1)p\Gamma_i(v^*)/2 > -1$, $0 \leqslant i \leqslant r+1$. Thus we have $\Gamma_i(d\gamma) > -1$, $0 \leqslant i \leqslant r+1$, which implies that γ' is indeed integrable. From the definition of γ', we have $v^{(m-1)/2}(1-x^2)^{-1/2}\gamma'^{-1} \leqslant c$, $\alpha' v^{(m-1)/2}\gamma'^{-1} \leqslant c$ and $\beta'(v^*)^{-(m-1)p/2} \leqslant c\gamma'$. The last one is (3.7a) with $u = \alpha'^{-1}\gamma'$. We also have

$$u^{1-q}v^{(m-1)q/2}(\alpha'\sqrt{1-x^2})^{-q/2}\alpha'$$
$$= (\alpha' v^{(m-1)/2}\gamma'^{-1})^{q/2}(v^{(m-1)/2}(1-x^2)^{-1/2}\gamma'^{-1})^{q/2}\gamma' \leqslant c\gamma',$$

and

$$u^{1-q}v^{(m-1)q/2}\alpha' = (\alpha' v^{(m-1)/2}\gamma'^{-1})^q\gamma' \leqslant c\gamma';$$

thus both are integrable and (3.7b) follows. Therefore the only condition left is then (3.7c), which is the same as (3.12). ∎

Remark 3.1. It is not clear whether the conditions (3.11) are sharp, although from the results in the Jacobi distributions, conditions of this type are needed.

4. Mean Convergence of Interpolating Polynomials

We now consider the mean convergence of the interpolating polynomials. Let $s \geqslant 0$, $C^s[-1, 1] = C^s$ denote the space of s times continuously differentiable functions. Throughout this section we use φ to denote

$$\varphi(x) = \sqrt{1 - x^2}.$$

Our results are based on the following theorem, which gives the weighted L^p boundness of the Hermite interpolating polynomials defined at (2.4).

Theorem 4.1. *Let $m \geqslant 1$ and $0 < p < +\infty$. Let $d\alpha$, $d\beta$ be generalized Jacobi distributions with $d\alpha$ satisfying* (3.11) *and*

$$(\alpha'\varphi)^{-mp/2}\, \varphi^{tp}\beta' \in L^1, \tag{4.1}$$

where $t \geqslant 0$. Then for $f \in C^{m-1}$

$$\|H_{nm}(d\alpha, f)\|_{d\beta, p} \leqslant cn^t \sum_{j=0}^{m-1} \max_{1 \leqslant k \leqslant n} |\varphi(x_{kn})^j f^{(j)}(x_{kn})|/n^j \tag{4.2}$$

in particular,

$$\|H_{nm}(d\alpha, f)\|_{d\beta, p} \leqslant cn^t \sum_{j=0}^{m-1} \|\varphi^j f^{(j)}\|_\infty / n^j. \tag{4.3}$$

Proof. First let $1 < p < +\infty$. Since for every fixed $d > 0$, $n^{-1} \leqslant \varphi(x)$ on $[-1 + dn^{-2}, 1 - dn^{-2}]$, it follows from Lemma 2.3 that

$$\begin{aligned} \|H_{nm}(d\alpha, f)\|_{d\beta, p} &\leqslant c\, \|H_{nm}(d\alpha, f)\, \chi_{\Delta_n(d)}\|_{d\beta, p} \\ &\leqslant cn^t\, \|H_{nm}(d\alpha, f)\, \varphi^t\|_{d\beta, p}. \end{aligned}$$

We then apply Theorem 3.4 with $P = H_{nm}(d\alpha, f)$ and $\varphi^{tp}\beta'$ in place of β'. Since $\gamma' \in L^1$, it follows from (2.13), (2.14), and Lemma 3.1 that

$$\sum_{k=1}^{n} \lambda_n(d\gamma, x_{kn}) \leqslant c \sum_{k=1}^{n} \gamma_n'(x_{kn})[\alpha_n'(x_{kn})]^{-1}\, \lambda_{kn}(d\alpha) \leqslant c \int_{-1}^{1} d\gamma < +\infty.$$

Our theorem follows for $1 < p < +\infty$. For $0 < p \leqslant 1$, we use one technique in [15, p. 886]. Let

$$\beta'_* = \beta'[1 + \varphi^{(2-p)t}(\alpha'\varphi)^{m(p-2)/2}]^{-1}.$$

Then $\beta'_* \leqslant \beta'$ and $\beta'_* \leqslant \varphi^{(p-2)t}\beta'(\alpha'\varphi)^{-m(p-2)/2}$. Therefore $d\beta_*$ is a generalized Jacobi distribution and it satisfies $(\alpha'\varphi)^{-m}\,\varphi^{2t}\beta'_* \leqslant (\alpha'\varphi)^{-mp/2}\,\varphi^{tp}\beta' \in L^1$; that is, β'_* satisfies (4.1) with $p = 2$. Hence from (4.2),

$$\|H_{nm}(d\alpha, f)\|_{d\beta_*, 2} \leqslant cn^t \sum_{j=0}^{m-1} \max_{1 \leqslant k \leqslant n} |\varphi(x_{kn})^j f^{(j)}(x_{kn})|/n^j.$$

Since by the definition of β'_*,

$$\begin{aligned}(\beta'^{-1/2}_*\beta'^{1/p})^{2p(2-p)} &= \beta'[1 + \varphi^{(2-p)t}(\alpha'\varphi)^{m(p-2)/2}]^{p/(2-p)} \\ &\leqslant 2^{p/(2-p)}\beta'[1 + \varphi^{pt}(\alpha'\varphi)^{-mp/2}],\end{aligned}$$

and $\beta' \in L^1$ as $d\beta$ is a generalized Jacobi distribution, we have $\beta'^{-1/2}_*\beta'^{1/p} \in L^{2p/(2-p)}$. Therefore by the Hölder inequality

$$\|H_{nm}(d\alpha, f)\|_{d\beta, p} \leqslant \|H_{nm}(d\alpha, f)\|_{d\beta_*, 2}\, \|\beta'^{-1/2}_*\beta'^{1/p}\|_{2p/(2-p)},$$

our theorem follows for $0 < p \leqslant 1$ as well. ∎

Since the Hermite interpolating polynomials are defined by the values of function and its derivatives at interpolating knots, it is clearly unrealistic to expect that the right-hand side of (4.2) or (4.3) be replaced by L^p norm.

Let $E_n(f) = \inf_{P \in \Pi_n} \|f - P\|_\infty$. From [5, Theorem 2, p. 172] there exists a polynomial $R_n \in \Pi_n$, such that

$$\|f^{(j)} - R_n^{(j)}\|_\infty \leqslant cE_n(f^{(m-1)})/n^{m-1-j}, \qquad \forall f \in C^{m-1}, \tag{4.4}$$

for $0 \leqslant j \leqslant m-1$. We now prove our theorems on the mean convergence of the Hermite interpolating polynomials.

THEOREM 4.2. *Let $m \geqslant 1$, $0 < p < +\infty$. Let $d\alpha$ be a generalized Jacobi distribution satisfying* (3.11), *and $u \in GJ$ such that $u\varphi^{-jp} \in L^1$ for fixed j, $0 \leqslant j \leqslant m-1$. If*

$$(\alpha'\varphi)^{-mp/2}\,\varphi^{(m-j-1)p}u \in L^1, \tag{4.5}$$

then

$$\lim_{n\to\infty} \|H_{nm}^{(j)}(d\alpha, f) - f^{(j)}\|_{u,p} = 0, \qquad \forall f \in C^{m-1}. \tag{4.6}$$

Proof. Since $H_{nm}(d\alpha, f)$ is a projector from C^{m-1} to $\prod_{mn-1}$, by (4.4), we only need to estimate $\|H_{nm}^{(j)}(d\alpha, f-R_n)\|_{u,p}$. By applying the Bernstein–Markov inequality [16, Theorem 5] repeatedly, we have

$$\|H_{nm}^{(j)}(d\alpha, f-R_n)\|_{u,p} \leqslant cn^j \, \|H_{nm}(d\alpha, f-R_n)\, \varphi^{-j}\|_{u,p},$$

which, by (4.4) and Theorem 4.1 with $t=m-1$ and $\beta' = u\varphi^{-jp}$, is bounded by $cE_n(f^{(m-1)}) \to 0$. ∎

Our next theorem gives the order of convergence. The proof is similar to the one above. We take $t=0$ in applying Theorem 4.1.

THEOREM 4.3. *Let assumptions be the same as in Theorem* 4.2. *If*

$$(\alpha'\varphi)^{-mp/2}\, \varphi^{-jp} u \in L^1, \tag{4.7}$$

then

$$\|H_{nm}^{(j)}(d\alpha, f) - f^{(j)}\|_{u,p} \leqslant cE_n(f^{(m-1)})/n^{m-j-1}, \qquad \forall f \in C^{m-1}. \tag{4.8}$$

Remark 4.1. When $m=1$, Theorem 4.2 is proved by Nevai [15], where (4.5) is proved to be necessary as well. Note that when $m=1$, conditions (3.11) are satisfied by all generalized Jacobi distributions. For $m=2$, (4.5) for $j>0$ and (4.7) can be shown to be necessary by the results in Nevai and Xu [18], where Theorems 4.2 and 4.3 are also proved for $m=2$ and $\alpha' = \psi w$ with w being an integrable Jacobi weight function. For the Jacobi distributions, these two theorems are proved by Vértesi and Xu [25], where (4.5) and (4.7) are shown to be "almost" necessary for even integer m. We note that the method applied in these papers is different from ours. In the present generality, (4.5) for $j>0$ and (4.7) may still be necessary, but it seems to be a difficult task to prove it.

We now consider other interpolating polynomials. Since the approach is similar, we illustrate the idea with the Hermite–Fejér interpolating polynomials, $Q_{nm}(d\alpha, f)$, which are defined to be the unique polynomials in Π_{mn-1} such that

$$\begin{aligned} &Q_{nm}(d\alpha, f, x_{kn}(d\alpha)) = f(x_{kn}(d\alpha)), \qquad 1 \leqslant k \leqslant n \\ &Q_{nm}^{(j)}(d\alpha, f, x_{kn}(d\alpha)) = 0, \qquad 1 \leqslant k \leqslant n, \quad 1 \leqslant j \leqslant m-1. \end{aligned} \tag{4.9}$$

Unlike the Hermite inerpolating polynomials, $Q_{nm}(d\alpha, f)$ are not projection operators. Their convergence behavior is described by using the weighted modulus of continuity defined in [3],

$$\omega_\varphi(f, t) = \sup_{0<h\leqslant t} \|f(\cdot + h\varphi(\cdot)/2) - f(\cdot - h\varphi(\cdot)/2\|_\infty,$$

where if $x \pm h\varphi(x) \notin (-1, 1)$, the expression inside $\|\cdot\|_\infty$ is taken to be zero. It is known from [3, Theorem 7.2.1, p. 79 and Theorem 7.3.1, p. 84] that for the polynomial P_n of best approximation of $f \in C$,

$$\|f - P_n\|_\infty \leqslant c\omega_\varphi\left(f; \frac{1}{n}\right)$$

and

$$\|\varphi^j P_n^{(j)}\|_\infty \leqslant cn^j \omega_\varphi\left(f; \frac{1}{n}\right), \qquad j > 0.$$

THEOREM 4.4. *Let $m \geqslant 2$ and $0 < p < +\infty$. Let $d\alpha$ be a generalized Jacobi distribution satisfying* (3.11) *and $u \in GJ$ be integrable. If*

$$(\alpha'\varphi)^{-mp/2} u \in L^1 \tag{4.10}$$

then

$$\|Q_{nm}(d\alpha, f) - f\|_{u,p} \leqslant c\omega_\varphi\left(f; \frac{1}{n}\right), \qquad \forall f \in C. \tag{4.11}$$

The proof of this theorem is similar to that of Theorem 4.2. Let P_n be the best approximation polynomial to f. By Theorem 4.1 with $t = 0$, we can obtain

$$\begin{aligned}\|Q_{nm}(d\alpha, P_n) - H_{nm}(d\alpha, P_n)\|_{u,p} &\leqslant c \sum_{j=1}^{m-1} \|\varphi^j P_n^{(j)}\|_\infty / n^j \\ &\leqslant c\omega_\varphi\left(f, \frac{1}{n}\right).\end{aligned}$$

Since $H_{nm}(d\alpha, P_n) = P_n$, our theorem follows easily from the triangle inequality. We leave the detail to the reader.

Remark 4.2. For $m = 2$, it follows from the general results of Máté and Nevai [7] that the condition (4.9) is also necessary for (4.10). For $m = 2$ and $\alpha' = \psi w$ with w an integrable Jacobi weight, Theorem 4.4 has been proved by Vértesi and Xu [26]. If we want $\lim_{n \to \infty} \|Q_{nm}(d\alpha, f) - f\|_{u,p} = 0$ instead of (4.10), the condition (4.9) might be relaxed to $(\alpha'\varphi)^{-mp/2} \varphi^{(m-1)p} u \in L^1$. For $m = 2$ and $\alpha' = \psi w$ with w being an integrable Jacobi weight function, this is indeed true as shown by Nevai and Vértesi [17]. For the generalized Jacobi distributions, it is not clear whether it will follow from our theorems. For the corresponding results for the Jacobi distributions, see [21, 24, 27].

We may apply our methods to other interpolating polynomials, for example, those considered in [27]. Since the application is straightforward, we shall not elaborate any further. We close this paper with the following remark. The method we presented in this paper leads to the results of the general nature, but the condition (3.11) may not be sharp. The mean convergence of the Hermite interpolating polynomials based on the Jacobi distribution ($\Gamma_i(d\alpha)=0$, $1 \leqslant i \leqslant r$) is proved in [25] under the following conditions. If m is odd, $\Gamma_i(d\alpha) \geqslant -\frac{1}{2}-2/m$, $i=0$, $r+1$, and if m is even, $\Gamma_i(d\alpha) \geqslant -\frac{1}{2}-1/m$ or $\Gamma_i(d\alpha) \geqslant -\frac{1}{2}-2/m$, $i=0$, $r+1$, and $\max\{\Gamma_0(d\alpha), \Gamma_{r+1}(d\alpha)\} - \min\{\Gamma_0(d\alpha), \Gamma_{r+1}(d\alpha)\} \leqslant 2/m$. These conditions are less restrictive than those of (3.11). Vértesi [24] proved that these conditions are almost (with the difference of $>$ and $\geqslant$) sharp. Thus, the conditions of this type are necessary in our theorem, but it shows that (3.11) is not sharp. See also Remark 3.1. Thus the question is whether (3.11) can be improved or it reflects the limitation of the method.

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